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Linear Hamiltonian Difference Systems: Disconjugacy and Jacobi-Type Conditions

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We consider a linear Hamiltonian Difference System for the so-called singular case so that discrete Sturm–Liouville Equations of higher order are included in our theory. We introduce the concepts of focal points for matrix-valued and generalized zeros for vector-valued solutions of the system and define disconjugacy for linear Hamiltonian Difference Systems. We prove a Reid Roundabout Theorem which gives conditions equivalent to positive definiteness of a certain discrete quadratic functional, among them the strengthened Jacobi's Condition and a condition on a certain Riccati Difference Equation. The key to this theorem is a discrete version of Picone's Identity. Furthermore, for the sake of generalization of our theorem, we introduce controllability for linear Hamiltonian Difference Systems and prove a Reid Roundabout Theorem for a more general functional and more general boundary conditions.

1. INTRODUCTION

In his landmark paper on difference equations [20], P. Hartman gave a definition for a generalized zero of a solution of an nth order difference equation and introduced his concept of disconjugacy for such equations. This definition has been employed in numerous works on difference equations (see, for example, [13, 19, 26]) and the following definition is in the spirit of Hartman's concept (see also [1, Definition 6.16.1; 6, 30]):

If the solution y of a scalar self-adjoint difference equation of order $2n$ is defined on $[0, N+2n] \cap \mathbb{Z}$, then 0 is said to be a generalized zero of y with order n only if $y_0 = y_1 = \cdots = y_{n-1} = 0$; and y has a generalized zero of order n at $k+1 \in [1, N+n+1] \cap \mathbb{Z}$, provided $y_k \neq 0$, $y_{k+1} = y_{k+2} = \cdots = y_{k+n-1} = 0$, and $(-1)^n y_k y_{k+n} \geq 0$ hold. The difference equation is called

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disconjugate on $[0, N]$, *if no solution has two or more generalized zeros of order* n *in* $[0, N+n+1]$.

Recently (see [6]), C. Ahlbrandt and A. Peterson proved that disconjugacy in the above sense implies the positive definiteness of a certain discrete quadratic functional. However, the equivalence of those two conditions still remained an open question.

Now, as is well known, such discrete Sturm-Liouville Equations are equivalent to certain linear Hamiltonian Difference Systems of the form

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{H}$$

where $x_k, u_k \in \mathbb{R}^n$ for $k \in [0, N+1] \cap \mathbb{Z}$, where A_k, B_k, C_k are $n \times n$-matrices for $k \in [0, N] \cap \mathbb{Z}$, and where the forward difference operator Δ is defined by $\Delta x_k := x_{k+1} - x_k$ for $k \in [0, N] \cap \mathbb{Z}$. To be more precise, for a special choice of the occurring matrices we have that any solution of the system (H) can be expressed in terms of a solution of the discrete Sturm-Liouville Equation and vice versa (see Lemma 2 in [9]). Furthermore, all the matrices B_k have rank 1 in this special case.

In the present paper we discuss general systems of the form (H) for the *singular* case which means that we allow the matrices B_k to be singular. One reason we do this is that we wish to cover the important case of discrete Sturm-Liouville Equations of order $2n$ with $n > 1$. Systems of the form (H) were introduced by L. Erbe and P. Yan [14] and have been studied for the *non-singular* case by L. Erbe and P. Yan [14–17], A. Peterson [27], C. Ahlbrandt [3], O. Došlý [12], and C. Ahlbrandt and A. Peterson [6]. These authors defined disconjugacy for such Hamiltonian Systems and proved that disconjugacy is equivalent to the fact that

$$\sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\} > 0$$

whenever $\Delta x_k = A_k x_{k+1} + B_k u_k$ holds for all $k \in [0, N] \cap \mathbb{Z}$ with $\sum_{k=0}^{N+1} x_k^2 > 0$ and $x_0 = x_{N+1} = 0$. Also, several other conditions equivalent to disconjugacy are given in those papers, among them for example conditions on the principal solution and on certain Riccati Matrix Difference Equations. It was C. Ahlbrandt who first used the term ‘‘Reid Roundabout Theorem’’ for a result which gives conditions equivalent to the strengthened Jacobi's Condition (see, for example, [3, 4]). For the ‘‘continuous’’ theory, we refer the reader to the books by W. T. Reid [33, Chap. VII; 34, Theorem V.6.3], and for a generalization to more general boundary conditions as presented in our Theorem 3 below, we refer to the book by W. Kratz [24, Theorem 2.4.1].

It follows from the theory for the non-singular case that a Reid Roundabout Theorem is known for example for the case of a second order linear self-adjoint difference equation $\Delta(r_k \Delta x_k) + p_k x_{k+1} = 0$ where each r_k is real and positive (see also [11, 18, 21, 22, 32]). The previous literature also covers the case of a self-adjoint vector difference equation of the form $\Delta(P_k \Delta x_k) + Q_k x_{k+1} = 0$ where each P_k is a positive definite matrix. This latter case has been studied by many authors, among them C. Ahlbrandt and J. W. Hooker [5], S. Chen and L. Erbe [10], A. Peterson and J. Ridenhour [28, 29], T. Peil and A. Peterson [25], and C. Ahlbrandt [2].

However, a theory for linear Hamiltonian Difference Systems including a Reid Roundabout Theorem without the non-singularity assumption on the matrices B_k has not been developed. It is the purpose of this paper to provide a theory without the non-singularity assumption, to give a definition of *disconjugacy* for this general case, and to state and prove a *Reid Roundabout Theorem*, containing a condition on focal points of the principal solution of (H) as well as a condition on a certain Riccati Equation (see Theorem 2 below). Of course, the case of Sturm−Liouville Equations will then be *included* in our theory and our concept of disconjugacy coincides with the definition given at the beginning of this introduction, but, although it is a generalization, it reads much smoother (compare Definition 4 below):

Let $k \in [1, N+1] \cap \mathbb{Z}$. *We say that a solution* (x, u) *of* (H) *has a generalized zero in* $(k-1, k]$ *provided* $x_{k-1} \neq 0$, $(I - A_{k-1})x_k = B_{k-1}c$ *for some* $c \in \mathbb{R}^n$, *and* $x_{k-1}^T c \leq 0$ *hold. System* (H) *is called disconjugate on* $(0, N+1]$ *if every solution* (x, u) *of* (H) *with* $x_0 = 0$ *has no generalized zeros in* $(0, N+1]$ *and if every solution of* (H) *has at most one generalized zero in* $(0, N+1]$.

Our new theory can be applied for example to the theory of discrete Sturm−Liouville Equations, where the B_k are singular; to Robust Control Theory, where discrete Riccati Equations with singular B_k show up (see [4]); and to discrete control theory, where our functional appears as the second variation of the problems (see, for example, [23, Chap. 8]).

Furthermore, besides giving a Reid Roundabout Theorem for boundary conditions $x_0 = x_{N+1} = 0$ and for the above given functional, we present an extension to a more general functional and to more general boundary conditions. Here, we need the additional assumption of *controllability* of the system (H), but systems with invertible B_k or systems equivalent to Sturm−Liouville Equations are easily seen to be controllable in our sense so that this extended Reid Roundabout Theorem applies to these problems also.

Besides solving an open question (C. Ahlbrandt writes in [3] on p. 515: "An open question is that of existence of a Reid Roundabout Theorem for systems which allows B_n to be singular"), the present paper introduces the notions of controllability, disconjugacy, generalized zeros, and focal points. The author believes that these concepts are essential in the further study of linear Hamiltonian Difference Systems and to derive results similar to the continuous theory as presented in the book by W. Kratz [24]. There, one key to the theory is *Picone's Identity* (see [31; 7, Proposition 6.1; 24, Chaps. 1.2 and 1.3]), and the key to our Reid Roundabout Theorem is a discrete version of Picone's Identity.

Finally, let us briefly summarize the set up of this paper. Well-known results related to the study of discrete Hamiltonian Difference Systems are given in the next section. Section 3 contains a derivation of the discrete version of Picone's Identity. In Section 4, focal points of $n \times n$-matrix-valued solutions of (H) and generalized zeros of vector-valued solutions of (H) are defined. The Reid Roundabout Theorem including an extended Riccati Equivalence is proved, but only for the boundary conditions $x_0 = x_{N+1} = 0$ and when the functional has the form $\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}$. In order to arrive at a theory for more general functionals and more general boundary conditions we introduce in Section 5 another system (H^*) and apply the results from Sections 2–4 to this system. Finally we introduce our concept of controllability which serves to prove the main result of this paper, Theorem 3.

2. PRELIMINARIES ON LINEAR HAMILTONIAN DIFFERENCE SYSTEMS

Let $n, N \in \mathbb{N}$ and $J := [0, N] \cap \mathbb{Z}$, $J^* := [0, N+1] \cap \mathbb{Z}$. For a vector- or matrix-valued function f defined on J^* we write $f_k := f(k)$, $k \in J^*$, and the forward difference operator Δ is defined by $\Delta f_k := f_{k+1} - f_k$, $k \in J$, while the shift operator E is given by $Ef_k := f_{k+1}$, $k \in J$ (compare [23, Chap. 2]). When writing f^T, $\operatorname{Ker} f$, $\operatorname{Im} f$, or f^{-1}, we always mean the vector- or matrix-valued function given by $f_k^T = \{f(k)\}^T$, the set-valued functions given by $\operatorname{Ker} f_k = \operatorname{Ker}\{f(k)\}$, and $\operatorname{Im} f_k = \operatorname{Im}\{f(k)\}$ (where Ker and Im denote the kernel and the image of a matrix), and the matrix-valued function given by $f_k^{-1} = \{f(k)\}^{-1}$, $k \in J^*$, respectively. Equations involving such functions are always meant to hold on the maximal set of definition, i.e., $\operatorname{Ker} Ef \subset \operatorname{Ker} f$ is an abbreviation for $\operatorname{Ker} f_{k+1} \subset \operatorname{Ker} f_k$ $\forall$ $k \in J$ whereas $f = 0$ stands for $f_k = 0$ $\forall$ $k \in J^*$. By $M^\dagger$ we denote the Moore–Penrose Inverse of the matrix M, i.e., the unique matrix satisfying $MM^\dagger M = M$ and $M^\dagger M M^\dagger = M^\dagger$ such that both $MM^\dagger$ and $M^\dagger M$ are symmetric (see, for example, [8, Theorem 1.5]). Finally, we write $D > 0$ if D is a positive definite matrix, and $D \geq 0$ if D is positive semidefinite.

Now, given $n \times n$-matrix-valued functions A, B, and C on J we require throughout that they satisfy the following general hypothesis on J:

$$B \text{ and } C \text{ are symmetric}, \qquad \tilde{A} := (I - A)^{-1} \text{ exists}.$$

The following is known as a *linear Hamiltonian Difference System* (on J):

$$\Delta \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \begin{pmatrix} Ex \\ u \end{pmatrix}. \tag{H}$$

Vector-valued solutions (x, u) of (H) are denoted by small letters while we use capital letters for $n \times n$-matrix-valued solutions (X, U) of (H), i.e., X and U satisfy $\Delta X = AEX + BU$ and $\Delta U = CEX - A^T U$.

Remark 1 (*Linear Hamiltonian Difference System*). (i) $\Delta x = AEx + Bu$ is referred to as the equation of motion while we call $\Delta u = CEx - A^T u$ Euler's Equation.

(ii) The well-known Wronskian Identity states that, for any two matrix- or vector-valued solutions (f_1, g_1) and (f_2, g_2) of (H), $f_1^T g_2 - g_1^T f_2$ is a constant function on J^*.

(iii) Invertibility of $I - A$ ensures that (x, u) solves (H) iff $E\binom{x}{u} = T\binom{x}{u}$ holds, where

$$T := \begin{pmatrix} \tilde{A} & \tilde{A}B \\ C\tilde{A} & C\tilde{A}B + I - A^T \end{pmatrix}$$

is invertible. Thus, any initial value problem (i.e., (H) together with fixed prescribed values for x_m and u_m, $m \in J^*$) is uniquely solvable.

DEFINITION 1 (*Conjoined Basis*). (i) If (X, U) solves (H) such that $X^T U$ is symmetric and $\operatorname{rank}(X^T \ U^T) = n$, then (X, U) is called a *conjoined basis* of (H).

(ii) Two conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ are called *normalized conjoined bases* of (H) if $X^T \tilde{U} - U^T \tilde{X} = I$ holds.

(iii) The solutions (X, U) and $(\tilde{X}, \tilde{U})$ of (H) satisfying the initial conditions $X_0 = \tilde{U}_0 = 0$ and $U_0 = -\tilde{X}_0 = I$ are called the *special normalized conjoined bases* of (H) (at 0). We call (X, U) the *principal solution* and $(\tilde{X}, \tilde{U})$ the *associated solution* of (H) (at 0).

Remark 2. (i) It may be readily verified that for any normalized conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of (H) we have (see, for example, [24, Proposition 1.1.5])

$$\begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{U}^T & -\tilde{X}^T \\ -U^T & X^T \end{pmatrix}$$

so that all of the following identities, which will be used frequently, hold:

$$X\tilde{U}^T - \tilde{X}U^T = \tilde{U}X^T - U\tilde{X}^T = X^T\tilde{U} - U^T\tilde{X} = \tilde{U}^TX - \tilde{X}^TU = I,$$

$$X^TU - U^TX = \tilde{X}^T\tilde{U} - \tilde{U}^T\tilde{X} = X\tilde{X}^T - \tilde{X}X^T = U\tilde{U}^T - \tilde{U}U^T = 0,$$

$$(E\tilde{X})X^T - (EX)\tilde{X}^T = \tilde{A}B, \quad (E\tilde{U})X^T - (EU)\tilde{X}^T = C\tilde{A}B + I - A^T,$$

$$(EX)\tilde{U}^T - (E\tilde{X})U^T = \tilde{A}, \qquad (EU)\tilde{U}^T - (E\tilde{U})U^T = C\tilde{A}.$$

(ii) Let (X, U) be a conjoined basis of (H) satisfying $\operatorname{Ker} EX \subset \operatorname{Ker} X$. Then $\operatorname{Ker} EX^T \subset \operatorname{Ker} B\tilde{A}^T$ holds. To see this, let $k \in J$ and let $c \in \operatorname{Ker} X_{k+1}^T$. Then $X_{k+1}\tilde{X}_{k+1}^T c = 0$ follows and hence

$$0 = X_k \tilde{X}_{k+1}^T c = \left(\tilde{X}_k X_{k+1}^T + B_k \tilde{A}_k^T\right) c = B_k \tilde{A}_k^T c.$$

(iii) Suppose that F and V are matrices with $VFV = V$ and VF symmetric. Then we have for any matrix W that $\operatorname{Ker} V \subset \operatorname{Ker} W$ iff $W = WFV$. To see this, we note that one direction of the claim is trivial and suppose now that $\operatorname{Ker} V \subset \operatorname{Ker} W$ holds. Then, for arbitrary c such that W^Tc is defined, there exists $d = Vd_1 + d_2$ with $d_2 \in \operatorname{Ker} V^T$ and

$$\begin{aligned} W^Tc &= V^Td = V^TVd_1 = V^TVFVd_1 = V^T(VF)^TVd_1 \\ &= V^TF^TV^TVd_1 = V^TF^TW^Tc = (WFV)^Tc \end{aligned}$$

so that $W = WFV$ follows.

3. DISCRETE QUADRATIC FUNCTIONALS AND PICONE'S IDENTITY

DEFINITION 2. Let be given $2n \times 2n$-matrices R and S with S symmetric.

(i) The *discrete quadratic functional* $\mathscr{F}$ is defined by

$$\mathscr{F}(x,u) := \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}.$$

(ii) (x,u) is called *admissible* if it satisfies the equation of motion.

(iii) If $\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} R^T$, then (x,u) is said to satisfy the *boundary conditions* given by R (we write $(x,u) \in \mathscr{R}$).

(iv) $\mathscr{F}$ is called *positive definite* (we write $\mathscr{F} > 0$) if $\mathscr{F}(x,u) > 0$ holds for all admissible $(x,u) \in \mathscr{R}$ with $x \neq 0$.

Remark 3. (i) In this and in the next section we put $R = S = 0$ which means that we will be dealing with the "classical" boundary conditions $x_0 = x_{N+1} = 0$. The general case is treated in Section 5.

(ii) Define the following (transition and controllability) matrices:

$$\begin{cases} \Phi_{km} := \tilde{A}_{k-1} \cdot \tilde{A}_{k-2} \cdot, \dots, \cdot \tilde{A}_m \qquad \text{for } k > m, \qquad \Phi_k := \Phi_{k0}, \\ G_k := (\Phi_{k0} B_0 \; \Phi_{k1} B_1 \; \cdots \; \Phi_{k,k-1} B_{k-1}) \qquad \text{for } k \in J^* \setminus \{0\}. \end{cases} \tag{+}$$

Then $x_k = \Phi_k x_0 + G_k \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \end{pmatrix}$ holds for admissible (x,u).

(iii) Suppose that $\Delta X = AEX + BU$ holds with $\operatorname{Ker} EX \subset \operatorname{Ker} X$ and that (x,u) is admissible. Then, if $x_k \in \operatorname{Im} X_k$ for some $k \in J$ we have that $x_{k+1} \in \operatorname{Im} X_{k+1}$. To see this, suppose $x_k = X_k c \in \operatorname{Im} X_k$. Then

$$x_{k+1} = \tilde{A}_k X_k c + \tilde{A}_k B_k u_k = X_{k+1} c + \tilde{A}_k B_k (u_k - U_k c)$$

so that $x_{k+1} \in \operatorname{Im} X_{k+1}$ by Remark 2(ii).

The following formula on how to compute $\mathscr{F}(x,u)$ is very useful, especially if (x,u) is a solution of (H).

LEMMA 1. *Let (x,u) be admissible. Then we have*

$$\sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} = \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} + \sum_{k=0}^{N} x_{k+1}^T \{C_k x_{k+1} - A_k^T u_k - \Delta u_k\}.$$

Proof. We apply the discrete product rule to see that

$$\begin{aligned}
&\sum_{k=0}^{N} x_{k+1}^T\{C_k x_{k+1} - A_k^T u_k - \Delta u_k\} \\
&\quad = \sum_{k=0}^{N} \{x_{k+1}^T(C_k x_{k+1} - A_k^T u_k) + (\Delta x_k^T)u_k - \Delta(x_k^T u_k)\} \\
&\quad = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} - x_{N+1}^T u_{N+1} + x_0^T u_0. \qquad \blacksquare
\end{aligned}$$

In the next lemma we collect some formulae that will be used in the remaining sections. Picone's Identity will be an easy consequence of these formulae.

LEMMA 2. *For $k \in J$ and an $n \times n$-matrix-valued Q on J^* we put*

$$R_k[Q] := \tilde{A}_k^T(Q_{k+1} - C_k)\tilde{A}_k(I + B_k Q_k) - Q_k.$$

(i) *For admissible (x, u) and symmetric Q,*

$$\begin{aligned}
&\Delta\{x^T Q x\} - (Ex)^T C(Ex) - u^T B u + z^T D z \\
&\qquad = 2u^T BR[Q]x + x^T\{R^T[Q] - QBR[Q]\}x
\end{aligned}$$

holds, where $z := u - Qx$ and $D := B - B\tilde{A}^T(EQ - C)\tilde{A}B$. Also, we have

$$\{I - A - B\tilde{A}^T(EQ - C)\}Ex = x + Dz - BR[Q]x.$$

(ii) *If (X, U) is a conjoined basis of* (H) *with* $\operatorname{Ker} EX \subset \operatorname{Ker} X$, *then we have for symmetric Q with $QX = UX^\dagger X$ that*

$$R[Q]X = 0$$

and $X(EX)^\dagger\tilde{A}B = B - B\tilde{A}^T(EQ - C)\tilde{A}B$ is symmetric. Furthermore,

$$Q := UX^\dagger + (UX^\dagger\tilde{X} - \tilde{U})(I - X^\dagger X)U^T$$

satisfies the above assumptions whenever (X, U) and $(\tilde{X}, \tilde{U})$ are normalized conjoined basis of (H).

Proof. To show (i), let $\tilde{Q} := \tilde{A}^T(EQ - C)\tilde{A}$. Then

$$\begin{aligned}
\Delta\{x^TQx\} &- (Ex)^T C(Ex) - u^TBu + z^TDz \\
&= (x + Bu)^T\tilde{Q}(x + Bu) - x^TQx - u^TBu \\
&\quad + (u - Qx)^T\big(B - B\tilde{Q}B\big)(u - Qx) \\
&= x^T\big\{\tilde{Q} - Q + Q\big(B - B\tilde{Q}B\big)Q\big\}x + 2u^T\big\{B\tilde{Q} - \big(B - B\tilde{Q}B\big)Q\big\}x \\
&= x^T\{R^T[Q] - QBR[Q]\}x + 2u^TBR[Q]x,
\end{aligned}$$

since $R[Q] = \tilde{Q}(I + BQ) - Q$. Finally, we have

$$\begin{aligned}
\big\{I - A - B\tilde{A}^T(EQ - C)\big\}Ex \\
&= x + Bu - B\tilde{Q}x - B\tilde{Q}Bu \\
&= x + Bu - B\tilde{Q}Bu - BR[Q]x - BQx + B\tilde{Q}BQx \\
&= x + D(u - Qx) - BR[Q]x.
\end{aligned}$$

Next, we prove (ii). We have, since $(EX)^\dagger(EX)X^\dagger = X^\dagger$ holds (compare Remark 2(iii)),

$$\begin{aligned}
R[Q]X &= \tilde{Q}(X + BU)X^\dagger X - UX^\dagger X \\
&= \tilde{A}^T(EQ - C)(EX)X^\dagger X - UX^\dagger X \\
&= \tilde{A}^T(EU)(EX)^\dagger(EX)X^\dagger X - \tilde{A}^TC(EX)X^\dagger X - UX^\dagger X \\
&= \tilde{A}^T\{EU - CEX - (I - A^T)U\}X^\dagger X,
\end{aligned}$$

and, because of $(EX)(EX)^\dagger\tilde{A}B = \tilde{A}B$ (compare Remark 2(ii) and (iii)),

$$\begin{aligned}
B - B\tilde{A}^T(EQ - C)\tilde{A}B \\
&= B - B\tilde{A}^T(EQ - C)(EX)(EX)^\dagger\tilde{A}B \\
&= B - B\tilde{A}^T(EU - CEX)(EX)^\dagger\tilde{A}B \\
&= B - BU(EX)^\dagger\tilde{A}B \\
&= \{(I - A)EX - BU\}(EX)^\dagger\tilde{A}B \\
&= X(EX)^\dagger\tilde{A}B.
\end{aligned}$$

The remainder of the assertions follows from the formula

$$UX^{\dagger} + (UX^{\dagger}\tilde{X} - \tilde{U})(I - X^{\dagger}X)U^T$$
$$= UX^{\dagger}X\tilde{U}^T + \tilde{U}X^{\dagger}XU^T - \tilde{U}U^T - UX^{\dagger}\tilde{X}X^{\dagger}XU^T. \qquad \blacksquare$$

THEOREM 1 (Picone's Identity). *Suppose* (X, U) *is a conjoined basis of* (H) *with* $\operatorname{Ker} EX \subset \operatorname{Ker} X$. *Then we have for admissible* (x, u) *with* $x_0 \in \operatorname{Im} X_0$

$$\sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} = x_{N+1}^T Q_{N+1} x_{N+1} - x_0^T Q_0 x_0 + \sum_{k=0}^{N} z_k^T D_k z_k,$$

where $z = u - (UX^{\dagger})^T x$ *and* $D = X(EX)^{\dagger}\tilde{A}B$ *is symmetric*. Q *may be any matrix satisfying the condition in Lemma* 2(ii).

Proof. This is a consequence of Lemma 2 and Remark 3(iii). $\blacksquare$

In view of Picone's Identity and Definition 2(iii), we define the following:

DEFINITION 3 (*Focal Points*). Let (X, U) be a conjoined basis of (H) and let $k \in J^* \setminus \{0\}$. k is called a *focal point* of X (or of (X, U)) if $\operatorname{Ker} X_k \not\subset \operatorname{Ker} X_{k-1}$. We say that X has a focal point in $(k-1, k)$ provided $\operatorname{Ker} X_k \subset \operatorname{Ker} X_{k-1}$ and $X_{k-1}X_k^{\dagger}\tilde{A}_{k-1}B_{k-1} \not\geq 0$.

Now, using this definition, we can prove a part of our Reid Roundabout Theorem which is not needed later on for its proof but which is more specific.

PROPOSITION 1. *If the principal solution* (X, U) *of* (H) *has a focal point in* $(0, N+1]$, *then* $\mathscr{F}$ *is not positive definite*. *More precisely we have that*

(i) $\operatorname{Ker} X_{m+1} \not\subset \operatorname{Ker} X_m$ *for some* $m \in J$ *implies the existence of an admissible* $(x, u) \in \mathscr{R}$ *with* $x \neq 0$ *and* $\mathscr{F}(x, u) = 0$;

(ii) $\operatorname{Ker} EX \subset \operatorname{Ker} X$ *and* $c^T X_m X_{m+1}^{\dagger}\tilde{A}_m B_m c < 0$ *for some* $m \in J \setminus \{0\}$ *and some* $c \in \mathbb{R}^n$ *implies the existence of an admissible* $(x, u) \in \mathscr{R}$ *with* $x \neq 0$ *and* $\mathscr{F}(x, u) < 0$.

Proof. First, if there exists $c \in \operatorname{Ker} X_{m+1} \setminus \operatorname{Ker} X_m$, we put

$$x_k := \begin{cases} X_k c, & 0 \leq k \leq m \\ 0, & m+1 \leq k \leq N+1, \end{cases}$$

$$u_k := \begin{cases} U_k c, & 0 \leq k \leq m \\ 0, & m+1 \leq k \leq N+1. \end{cases}$$

Then $x \neq 0$ since $x_m \neq 0$ and $(x, u) \in \mathscr{R}$ because of $x_0 = x_{N+1} = 0$. We have

$$\tilde{A}_m(x_m + B_m u_m) = X_{m+1}c = 0 = x_{m+1}$$

which implies that (x, u) is admissible, and Lemma 1 yields

$$\mathscr{F}(x, u) = \sum_{k=0}^{m-1} x_{k+1}^T\{C_k x_{k+1} - A_k^T u_k - \Delta u_k\} = 0.$$

Next, suppose $\operatorname{Ker} EX \subset \operatorname{Ker} X$ holds. Let $c \in \mathbb{R}^n$ with $c^T D_m c < 0$, where we put $D_m = X_m X_{m+1}^\dagger \tilde{A}_m B_m$. Let $d := -X_{m+1}^\dagger \tilde{A}_m B_m c$ and define

$$x_k := \begin{cases} X_k d, & 0 \le k \le m-1 \\ -D_m c, & k = m \\ 0, & m+1 \le k \le N+1, \end{cases}$$

$$u_k := \begin{cases} U_k d, & 0 \le k \le m-1 \\ \tilde{A}_m^T\left(X_{m+1}^\dagger\right)^T X_m^T c, & k = m \\ 0, & m+1 \le k \le N+1. \end{cases}$$

Again we have $x \neq 0$ because of $x_m \neq 0$ and $x_0 = x_{N+1} = 0$ yields $(x, u) \in \mathscr{R}$. To see that (x, u) is admissible, we note that

$$\begin{aligned} \tilde{A}_{m-1}(x_{m-1} + B_{m-1}u_{m-1}) &= X_m d = -D_m c = x_m, \\ \tilde{A}_m(x_m + B_m u_m) &= \tilde{A}_m\left(-D_m + D_m^T\right)c = 0 = x_{m+1} \end{aligned}$$

hold since D_m is symmetric. Now, Lemma 1 shows that

$$\begin{aligned} \mathscr{F}(x, u) &= \sum_{k=0}^{m-1} x_{k+1}^T\left\{C_k x_{k+1} + \left(I - A_k^T\right)u_k - u_{k+1}\right\} \\ &= d^T X_m^T\left\{C_{m-1}X_m d + \left(I - A_{m-1}^T\right)U_{m-1}d - \tilde{A}_m^T\left(X_{m+1}^\dagger\right)^T X_m^T c\right\} \\ &= d^T X_m^T U_m d - d^T X_m^T c + d^T U_m^T D_m^T c = c^T D_m c < 0. \qquad \blacksquare \end{aligned}$$

Remark 4. Of course, Picone's Identity together with the last part of Lemma 2(i) could now be used to show that the absence of focal points of the principal solution implies positive definiteness of the quadratic functional, so that the strengthened Jacobi's Condition holds. But we will do a detour and include a Riccati Equivalence since such an equivalence may be important for applications of our theory. Also, although we could define disconjugacy in terms of focal points of the principal solution (as is done in

[24, Definition 8.6.4]) we rather take the "classical" way and define disconjugacy in terms of generalized zeros of vector-valued solutions (x, u) of (H). This serves to compare our results easily with the previous literature, for example, in view of the Hartman-like definition at the beginning of Section 1. However, all these concepts will turn out to be equivalent, and this is shown in the next section.

4. DISCONJUGACY FOR LINEAR HAMILTONIAN DIFFERENCE SYSTEMS

PROPOSITION 2. *The principal solution* (X, U) *of* (H) *has no focal points in* $(0, N+1]$ *if and only if the following condition* $(*)$ *holds*:

$$\begin{cases} x_m^T B_m^\dagger (I - A_m) x_{m+1} > 0 \ \forall\ m \in J \text{ with } x_m \neq 0,\ x_{m+1} \in \operatorname{Im} \tilde{A}_m B_m \\ \text{for all solutions } (x, u) \text{ of } (\mathrm{H}) \text{ with } x_0 = 0. \end{cases} \tag{$*$}$$

Proof. Let $m \in J$. Suppose $\operatorname{Ker} EX \subset \operatorname{Ker} X$ and $D = X(EX)^\dagger \tilde{A} B \geq 0$. Let (x, u) solve (H) with $x_0 = 0$. By uniqueness of solutions of initial value problems (compare Remark 1(iii)) we know that $(x, u) = (Xu_0, Uu_0)$. Suppose $x_m \neq 0$ and $x_{m+1} = \tilde{A}_m B_m c \in \operatorname{Im} \tilde{A}_m B_m$. Then the following, where we use $x_m = X_m X_{m+1}^\dagger x_{m+1}$ (compare Remark 2(iii)), holds:

$$\begin{aligned} x_m^T B_m^\dagger (I - A_m) x_{m+1} &= x_{m+1}^T \left(X_{m+1}^\dagger\right)^T X_m^T B_m^\dagger (I - A_m) x_{m+1} \\ &= c^T D_m^T B_m^\dagger B_m c = c^T D_m c \geq 0, \end{aligned}$$

and $c^T D_m c = 0$ yields

$$0 = D_m c = X_m X_{m+1}^\dagger \tilde{A}_m B_m c = X_m X_{m+1}^\dagger x_{m+1} = x_m,$$

so that $x_m^T B_m^\dagger (I - A_m) x_{m+1} > 0$ holds.

Next, if $(*)$ holds, we let $\alpha \in \operatorname{Ker} X_{m+1}$ and define $(x, u) := (X\alpha, U\alpha)$, so that (x, u) solves (H) with $x_0 = 0$ and $x_{m+1} = 0 \in \operatorname{Im} \tilde{A}_m B_m$ which yields $x_m = 0$ too and thus $\operatorname{Ker} X_{m+1} \subset \operatorname{Ker} X_m$. To end the proof in this direction, let $c \in \mathbb{R}^n$ be arbitrary and put

$$(x, u) := (X\alpha, U\alpha) \qquad \text{with } \alpha := X_{m+1}^\dagger \tilde{A}_m B_m c.$$

Thus (x, u) solves (H) with $x_0 = 0$ and (use Remark 2(ii) and (iii)) $x_{m+1} = \tilde{A}_m B_m c \in \operatorname{Im} \tilde{A}_m B_m$. Therefore $0 < x_m^T B_m^\dagger (I - A_m) x_{m+1} = c^T D_m c$ is valid provided $D_m c = x_m \neq 0$ holds and we have $D_m \geq 0$. ∎

Remark 5 (*Sturm–Liouville Equations*). (i) The proof of the above proposition shows that if $\operatorname{Ker} EX \subset \operatorname{Ker} X$ holds for the principal solution (X, U) of (H), we have that

$$X(EX)^{\dagger}\tilde{A}B \geq 0 \qquad \text{iff} \qquad (I - A^T)B^{\dagger}X(EX)^{\dagger} \geq 0 \text{ on } \operatorname{Im} \tilde{A}B.$$

(ii) Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$-matrices with $b_{nn} = a_{i,i+1} = 1$ (and all other entries 0). The resulting Hamiltonian System is equivalent to a Sturm–Liouville Equation in the sense described in the Introduction and the relationship between solutions (x, u) of (H) and solutions y of the Sturm–Liouville Equation is given by

$$x_{k+n} = \begin{pmatrix} y_{k+n} & \Delta y_{k+n-1} & \Delta^2 y_{k+n-2} & \cdots & \Delta^{n-1} y_{k+1} \end{pmatrix}^T.$$

For a more precise discussion of this topic see [9, Lemma 2]. However, with the observations that

$$\tilde{A}B = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

$$x_{k+n} \in \operatorname{Im} \tilde{A}B \qquad \text{iff} \qquad y_{k+1} = y_{k+2} = \cdots = y_{k+n-1} = 0,$$

and $x_{k+n-1}^T B^{\dagger}(I - A)x_{k+n} = (-1)^{n-1} y_k y_{k+n}$ in this case of $x_{k+n} \in \operatorname{Im} \tilde{A}B$, one may easily see how the Hartman-like definition, used by C. Ahlbrandt and A. Peterson and given in the introduction of this paper, shows up in a natural way within the framework of our theory. This, of course, motivates the next definition.

DEFINITION 4 (*Generalized Zeros, Disconjugacy*). (i) Let (x, u) solve (H) and let $k \in J^* \setminus \{0\}$. We say that x (or (x, u)) has a *generalized zero* in $(k-1, k]$ provided $x_{k-1}^T B_{k-1}^{\dagger}(I - A_{k-1})x_k \leq 0$ holds with $x_{k-1} \neq 0$ and $x_k \in \operatorname{Im} \tilde{A}_{k-1} B_{k-1}$.

(ii) The system (H) is called *disconjugate* on J^* if no solution (x, u) of (H) with $x_0 = 0$ has one or more generalized zeros in $(0, N+1]$ and if no solution (x, u) of (H) with $x_0 \neq 0$ has two or more generalized zeros in $(0, N+1]$.

Remark 6. Let (x, u) solve (H) and suppose that $x_{k+1} = \tilde{A}_k B_k c$ for some $c \in \mathbb{R}^n$ and some $k \in J$. Then we have

$$B_k B_k^{\dagger} x_k = B_k B_k^{\dagger}(B_k c - B_k u_k) = B_k c - B_k u_k = x_k.$$

Thus, (H) is not disconjugate on J^* iff there exists a solution (x, u) of (H), integers $m, p \in J$, $m < p$, and vectors $c_m, c_p \in \mathbb{R}^n$ with

$$x_{m+1} = \tilde{A}_m B_m c_m, \qquad x_{p+1} = \tilde{A}_p B_p c_p, \qquad x_p \neq 0,$$
$$x_m^T c_m \leq 0, \qquad x_p^T c_p \leq 0.$$

PROPOSITION 3. *If $\mathscr{F}$ is positive definite, then* (H) *is disconjugate on J^*.*

Proof. We assume that (x, u) solves (H) and that there exist $m, p \in J$ with $m < p$ and $c_m, c_p \in \mathbb{R}^n$ such that $x_p \neq 0$ and $x_{k+1} = \tilde{A}_k B_k c_k$, $x_k^T c_k \leq 0$ hold for $k \in \{m, p\}$ (compare Remark 6). Now we define the following:

$$\tilde{x}_k := \begin{cases} x_k, & m+1 \leq k \leq p \\ 0, & \text{otherwise}, \end{cases} \qquad \tilde{u}_k := \begin{cases} c_m, & k = m \\ u_k, & m+1 \leq k \leq p-1 \\ u_p - c_p, & k = p \\ 0, & \text{otherwise}. \end{cases}$$

Thus, $\tilde{x}_0 = \tilde{x}_{N+1} = 0$, $\tilde{x}_p \neq 0$, and

$$\tilde{A}_m \tilde{x}_m + \tilde{A}_m B_m \tilde{u}_m = \tilde{A}_m B_m c_m = x_{m+1} = \tilde{x}_{m+1},$$

$$\tilde{A}_p \tilde{x}_p + \tilde{A}_p B_p \tilde{u}_p = \tilde{A}_p x_p + \tilde{A}_p B_p u_p - \tilde{A}_p B_p c_p = 0 = \tilde{x}_{p+1}$$

show admissibility of $(\tilde{x}, \tilde{u}) \in \mathscr{R}$. But

$$\begin{aligned} \mathscr{F}(\tilde{x}, \tilde{u}) &= \sum_{k=m}^{p-1} \tilde{x}_{k+1}^T \left\{ C_k \tilde{x}_{k+1} - A_k^T \tilde{u}_k - \Delta \tilde{u}_k \right\} \\ &= x_{m+1}^T \left(I - A_m^T \right) (c_m - u_m) + x_p^T c_p = c_m^T x_m + x_p^T c_p \leq 0, \end{aligned}$$

so that $\mathscr{F} \not> 0$ which proves the assertion. ∎

Now we are ready to prove our Reid Roundabout Theorem for the case of boundary conditions of the form $x_0 = x_{N+1} = 0$. For the Riccati Equivalence part we use the notation introduced in Lemma 2 and $(+)$ from Remark 3(ii).

THEOREM 2 (Reid Roundabout Theorem). *Suppose $R = S = 0$ (see Definition 2). Then the following statements are equivalent*:

(i) $\mathscr{F} > 0$;

(ii) *System* (H) *is disconjugate on J^**;

(iii) *The principal solution of* (H) *has no focal points in* $(0, N+1]$;

(iv) $R[Q]G = 0$ *has a symmetric solution* Q *with* $B - B\tilde{A}^T(EQ - C)\tilde{A}B \geq 0$, *where* $R[Q] = \tilde{A}^T(EQ - C)\tilde{A}(I + BQ) - Q$ *and* G *is defined by* $(+)$.

Proof. We split the proof into five parts according to the following figure, where condition $(*)$ is from Proposition 2.

$$\begin{tikzcd}
\boxed{\text{(i) } \mathscr{F} > 0} \arrow[r, Rightarrow, "(1)"] & \boxed{\text{(ii) Disconjugacy}} \arrow[r, Rightarrow, "(2)"] & \boxed{\text{Condition } (*)} \arrow[d, Rightarrow, "(3)"] \\
\boxed{\text{(iv) Riccati Condition}} \arrow[u, Rightarrow, "(5)"] & & \boxed{\text{(iii) No focal points}} \arrow[ll, Rightarrow, "(4)"']
\end{tikzcd}$$

While (1) and (3) already have been shown in Propositions 3 and 2, respectively, (2) is a trivial consequence of the definition of disconjugacy given in Definition 4(ii). We now employ Lemma 2 together with Remark 3(ii) to show the remaining two assertions.

To show (4), suppose that $\operatorname{Ker} EX \subset \operatorname{Ker} X$ and $D := X(EX)^\dagger\tilde{A}B \geq 0$ hold, where (X, U) is the principal solution of (H). With

$$Q := UX^\dagger + (UX^\dagger\tilde{X} - \tilde{U})(I - X^\dagger X)U^T$$

defined as in Lemma 2(ii) we have that Q is symmetric, that $B - B\tilde{A}^T(EQ - C)\tilde{A}B = D \geq 0$, and that $R_k[Q]X_k = 0$ holds for all $k \in J$. Now, if $x_0 = 0 \in \operatorname{Im} X_0$, we have by Remark 3(ii) and (iii) that

$$x_k := G_k \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \end{pmatrix} \in \operatorname{Im} X_k$$

holds for any choice of $u_0, \ldots, u_{k-1} \in \mathbb{R}^n$ which yields $R_k[Q]G_k = 0$.

For proving (5), let be given a symmetric solution Q of the equation $R[Q]G = 0$ with $D := B - B\tilde{A}^T(EQ - C)\tilde{A}B \geq 0$ and suppose that $(x, u) \in \mathscr{R}$ is admissible so that we have $x_0 = x_{N+1} = 0$ and

$$R_k[Q]x_k = R_k[Q]G_k \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \end{pmatrix} = 0 \qquad \forall\, k \in J.$$

Lemma 2(i) immediately shows (with $z = u - Qx$) that

$$\mathscr{F}(x,u) = \sum_{k=0}^{N} z_k^T D_k z_k \geq 0.$$

To prove positive definiteness, we assume that $\mathscr{F}(x,u)$ vanishes for some admissible (x,u) with $x_0 = x_{N+1} = 0$. But then $D_k z_k = 0$ for all $k \in J$, and the last part of Lemma 2(i) yields

$$\left\{I - A_k - B_k \tilde{A}_k^T (Q_{k+1} - C_k)\right\} x_{k+1} = x_k \qquad \forall\, k \in J,$$

so that, because of $x_{N+1} = 0$, we have $x_k = 0$ for all $k \in J^*$, i.e., $x = 0$. ∎

Remark 7 (*Riccati Equivalence*). (i) Let us make a few remarks concerning what we have called Riccati Equivalence, i.e., item (iv) of the above theorem. Suppose that G_κ has full rank for some $\kappa \in J^*$. Then, by the definition of G (see Remark 3(ii)), $\operatorname{rank} G_k = n$ for all $k \in J^*$ with $k \geq \kappa$. Now, let $k \in J^*$ with $k \geq \kappa$. Of course, $R_k[Q]G_k = 0$ is equivalent to $R_k[Q] = 0$ in this case. Assume furthermore that $I + B_k Q_k$ is invertible and then we have

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k (I + B_k Q_k)^{-1} (I - A_k).$$

Since, by putting $D_k := (I + B_k Q_k)^{-1} B_k$, this equation could as well be written as

$$\Delta Q_k = C_k - A_k^T Q_k - Q_k A_k + A_k^T Q_k A_k - \left(I - A_k^T\right) Q_k D_k Q_k (I - A_k),$$

one talks about a Riccati Matrix Difference Equation. Finally, it is well-known and readily verified that this equation has a symmetric solution if and only if there exists a conjoined basis (X,U) of (H) such that both X_k and X_{k+1} are invertible.

(ii) The main purpose of this paper is to present a Reid Roundabout Theorem for the general functional and boundary conditions suggested by Definition 2. While for Theorem 2 we only need the principal solution (X,U) of (H) it will turn out that for this purpose the associated solution of (H) is needed also. The key of our approach is, besides the ideas shown so far, the first lemma of the next section which leads to a generalization of Picone's Identity. Also, for the general Reid Roundabout Theorem a further controllability assumption is needed, and this concept is introduced in the next section.

5. THE REID ROUNDABOUT THEOREM

LEMMA 3. *Suppose that* (X, U) *and* $(\tilde{X}, \tilde{U})$ *are any two* (*matrix-valued*) *solutions of* (H). *Define* $2n \times 2n$*-matrix-valued functions as*

$$X^* := \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix}, \qquad U^* := \begin{pmatrix} I & 0 \\ U & \tilde{U} \end{pmatrix},$$

$$A^* := \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad B^* := \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \qquad C^* := \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}.$$

Then (X, U) *and* $(\tilde{X}, \tilde{U})$ *are normalized conjoined bases of* (H) *if and only if* (X^*, U^*) *is a conjoined basis of*

$$\Delta\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A^* & B^* \\ C^* & -A^{*T} \end{pmatrix}\begin{pmatrix} Ex \\ u \end{pmatrix}. \tag{H*}$$

Proof. This is easily verified by some computation, and in fact there is no real difference to the "continuous" version as stated in [24, Proposition 1.1.6]. ∎

Remark 8. In order to apply our theory as presented in Sections 2–4 to the "big" system (H^*) we need to provide some formulae concerning "big" Moore–Penrose Inverses etc., and this is done in the following, where (X, U) and $(\tilde{X}, \tilde{U})$ are normalized conjoined bases of (H). We use the notation introduced in Lemma 3 and put $D^* = X^*(EX^*)^\dagger \tilde{A}^* B^*$ and $D = X(EX)^\dagger \tilde{A} B$.

(i) The definition of the Moore–Penrose Inverse yields that there exists a unique F with $FXF = F$, $XFX = X$, such that FX and $XF(I + \tilde{X}\tilde{X}^T)$ are symmetric, and that the formula

$$X^{*\dagger} = \begin{pmatrix} -F\tilde{X} & F \\ (I + \tilde{X}^T\tilde{X})^{-1}(I + \tilde{X}^T XF\tilde{X}) & (I + \tilde{X}^T\tilde{X})^{-1}\tilde{X}^T(I - XF) \end{pmatrix}$$

holds. Now, since $\tilde{F} := X^\dagger XF$ also satisfies the above assumptions we have that $\tilde{F} = F$ and $FX = X^\dagger X$ yields

$$X^{*\dagger}X^* = \begin{pmatrix} X^\dagger X & 0 \\ 0 & I \end{pmatrix}.$$

(ii) Of course, $\operatorname{Ker} EX \subset \operatorname{Ker} X$ iff $\operatorname{Ker} EX^* \subset \operatorname{Ker} X^*$. In this case D^* is symmetric and Remark 2(ii) and (iii) yield that $B\tilde{A}^T = B\tilde{A}^T(EF)^T(EX)^T$ so that we have

$$X(EF)\tilde{A}B = X(EX)^\dagger(EX)(EF)\tilde{A}B = X(EX)^\dagger\tilde{A}B$$

and $D^* = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$. Thus $D^* \geq 0$ iff $D \geq 0$.

(iii) The following Q^* is symmetric and satisfies $Q^*X^* = U^*X^{*\dagger}X^*$:

$$Q^* := \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X & X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X)U^T \\ \left\{X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X)U^T\right\}^T & UX^\dagger + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X)U^T \end{pmatrix}.$$

If (x, u) is admissible for (H), then $(x^*, u^*) := \left(\binom{\alpha}{x}, \binom{0}{u}\right)$ is admissible for (H^*), where $\alpha \in \mathbb{R}^n$, and we have, if $x^* \in \operatorname{Im} X^*$,

$$(u^* - Q^* x^*)^T D^* = (0\ \ u^T)\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} - (\alpha^T\ \ x^T)\begin{pmatrix} 0 & X^\dagger D \\ 0 & UX^\dagger D \end{pmatrix}$$

$$= \left(0 \quad u - (UX^\dagger)^T x - (X^\dagger)^T \alpha\right)^T D.$$

PROPOSITION 4 (Generalization of Picone's Identity). *Suppose* (X, U) *and* $(\tilde{X}, \tilde{U})$ *are normalized conjoined basis of* (H) *with* $\operatorname{Ker} EX \subset \operatorname{Ker} X$. *Let* $\alpha \in \mathbb{R}^n$. *Then we have for admissible* (x, u) *with* $\alpha + U_0^T x_0 \in \operatorname{Im} X_0^T$ *that*

$$\sum_{k=0}^{N} \left\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\right\}$$

$$= \begin{pmatrix} \alpha \\ x_{N+1} \end{pmatrix}^T Q_{N+1}^* \begin{pmatrix} \alpha \\ x_{N+1} \end{pmatrix} - \begin{pmatrix} \alpha \\ x_0 \end{pmatrix}^T Q_0^* \begin{pmatrix} \alpha \\ x_0 \end{pmatrix} + \sum_{k=0}^{N} z_k^T D_k z_k,$$

where $z = u - (UX^\dagger)^T x - (X^\dagger)^T \alpha$ *and where* Q^* *and* D *are defined as in Remark* 8(iii).

Proof. This follows from Remark 8 and Picone's Identity, Theorem 1. ∎

Before stating and proving the main result of this paper, it is essential now to introduce our concept of controllability for linear Hamiltonian Difference Systems. Although Theorem 3 below could be proved under weaker controllability assumptions, we state the definition of controllability as follows:

DEFINITION 5 (*Controllability*). The system (H) is called *controllable* on J^* if there exists $k \in J^* \setminus \{0\}$ such that for any solution (x, u) of (H) and any $m \in J$ with $m + k \in J^*$, we have that $x_m = x_{m+1} = \cdots = x_{m+k} = 0$ implies that $x = u = 0$ holds on J^*. The minimal integer with this property is then called the controllability index of (H).

Remark 9. (i) Whenever B is invertible or whenever (H) is equivalent to a Sturm–Liouville Equation of order $2n$ (with $N \geq n$), (H) is controllable with controllability index 1 and n, respectively (compare [9, Remark 2(i)]).

(ii) Controllability with index κ is equivalent to the following property: For all $k \geq \kappa$ and for any $a, b \in \mathbb{R}^n$ and any $m \in J$ with $m + k \in J^*$, there exists an admissible (x, u) with $x_m = a$ and $x_{m+k} = b$. Also, the matrix G_k defined in Remark 3(ii) has full rank in this case (compare [9, Lemma 3]). When a time-invariant (or autonomous) system is given, i.e., $A_k \equiv A$ and $B_k \equiv B$ on J, then controllability on J^* is equivalent to $\operatorname{rank}(B\ \tilde{A}B\ \cdots\ \tilde{A}^N B) = n$.

(iii) If (H) is controllable on J^* and if (X, U) is the principal solution of (H), then the absence of focal points of X in $(0, N+1]$ implies invertibility of X_{N+1}. To see this, let $c \in \operatorname{Ker} X_{N+1}$ and define $(x, u) := (Xc, Uc)$ which, of course, solves (H) with $x_0 = x_1 = \cdots = x_{N+1} = 0$ because $\operatorname{Ker} EX \subset \operatorname{Ker} X$. Since (H) is controllable on J^*, $x = u = 0$ on J^* follows and $0 = u_0 = U_0 c = c$ holds, and so X_{N+1} is invertible.

(iv) For the proof of our Reid Roundabout Theorem below we essentially only need that X_{N+1} is invertible where (X, U) is the principal solution of (H). Controllability of (H) ensures this property as is shown in (iii). But for this in fact we only need "controllability at 0," i.e., the requirements of Definition 5 only for $m = 0$. Thus we could replace the controllability assumption in Theorem 3 below by the assumption that $\operatorname{rank} G_k = n$ for some $k \in J^* \setminus \{0\}$.

(v) Suppose that the principal solution (X, U) has no focal points in $(0, N+1]$ and that (H) is controllable on J^* with controllability index κ. Then $\operatorname{rank} G_k = \operatorname{rank} X_k = n$ by (ii) and (iii) and $Q_k := U_k X_k^{-1}$ is well-defined for $k \in [\kappa, N+1] \cap \mathbb{Z}$. Now, let $k \in [\kappa, N] \cap \mathbb{Z}$. From Remark 7(i) we have that Q solves $Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k)$ and that $(I + B_k Q_k)^{-1} B_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k \geq 0$. This is what L. Erbe and P. Yan state as a Conjecture in [17] so that, assuming controllability, their conjecture holds in fact on $[\kappa, N] \cap \mathbb{Z}$, and their assumption $B_k \geq 0$ is not even needed.

THEOREM 3 (Extended Reid Roundabout Theorem). *Suppose* (H) *is controllable on* J^*. *Then the following statements are equivalent*:

(i) $\mathcal{F} > 0$;

(ii) *System* (H) *is disconjugate on* J^* *and*

$$\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} > 0$$

for all solutions $(x, u) \in \mathcal{R}$ *of* (H) *with* $\begin{pmatrix} x_0 \\ x_{N+1} \end{pmatrix} \neq 0$;

(iii) *X has no focal points in* $(0, N+1]$ *and*

$$M := R\left\{S + \begin{pmatrix} -X_{N+1}^{-1}\tilde{X}_{N+1} & X_{N+1}^{-1} \\ \left(X_{N+1}^{-1}\right)^T & U_{N+1}X_{N+1}^{-1} \end{pmatrix}\right\}R^T > 0 \qquad \textit{on } \operatorname{Im} R,$$

where $(X, U), (\tilde{X}, \tilde{U})$ *are the special normalized conjoined bases of* (H);

(iv) $R^*[Q^*]\begin{pmatrix} -I & 0 \\ \Phi & G \end{pmatrix} = 0$ *has a symmetric solution* Q^* *with* $Q_0^* = 0$ *and*

$$D^* := B^* - B^*\tilde{A}^{*T}(EQ^* - C^*)\tilde{A}^*B^* \geq 0 \qquad \textit{and}$$

$$R(S + Q_{N+1}^*)R^T > 0 \qquad \textit{on } \operatorname{Im} R,$$

where $R^*[Q^*] = \tilde{A}^{*T}(EQ^* - C^*)\tilde{A}^*(I + B^*Q^*) - Q^*$ *and where* G *and* Φ *are defined by* $(+)$ *in Remark* 3(ii).

Proof. Let (X, U) and $(\tilde{X}, \tilde{U})$ denote the special normalized conjoined bases of (H). First of all note that, if (X, U) has no focal points in $(0, N+1]$, X_{N+1} is invertible due to Remark 9(iii) and that M in (iii) is then symmetric, so that all parts of the assertion make sense. Observe also that $M = R(S + U_{N+1}^* X_{N+1}^{*-1})R^T$ where X^* and U^* are defined according to Lemma 3. We need to show some relations in addition to the equivalences already proved in Theorem 2.

While (i) implies (ii) trivially because of Lemma 1, we now assume that (ii) holds. Pick an arbitrary $c \in \operatorname{Im} R \setminus \{0\}$ and define

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix} X_{N+1}^{*-1} R^T c.$$

Now, $\begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = U_{N+1}^* X_{N+1}^{*-1} R^T c$ and $\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} = R^T c \neq 0$. Since $(x, u) \in \mathscr{R}$ solves (H), statement (ii) yields

$$0 < \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = c^T M c,$$

and hence $M > 0$ on $\operatorname{Im} R$.

Now suppose that (iii) holds and define Q^* by

$$Q^* := \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X & X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X)U^T \\ \left\{X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X)U^T\right\}^T & UX^\dagger + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X)U^T \end{pmatrix}$$

as is done in Remark 8(iii). Lemma 2(ii), applied to the system (H^*), yields

$R_k^*[Q^*]X_k^* = 0$ for all $k \in J$. Since $x_0^* := \begin{pmatrix} -x_0 \\ x_0 \end{pmatrix} \in \operatorname{Im} X_0^*$ for any $x_0 \in \mathbb{R}^n$ and since $\operatorname{Ker} EX^* \subset \operatorname{Ker} X^*$ (compare Remark 3(ii), (iii) and Remark 8(ii), (iii)), we know that for any choice of $u_0, \dots, u_{k-1} \in \mathbb{R}^n$,

$$\begin{pmatrix} -I & 0 \\ \Phi_k & G_k \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \\ \vdots \\ u_{k-1} \end{pmatrix} =: x_k^* \in \operatorname{Im} X_k^*$$

holds so that $R_k^*[Q^*]\begin{pmatrix} -I & 0 \\ \Phi_k & G_k \end{pmatrix} = 0$ is valid for all $k \in J$. Furthermore, Q^* is symmetric, $Q_0^* = 0$, and

$$D^* = X^*(EX^*)^\dagger \tilde{A}^* B^* = \begin{pmatrix} 0 & 0 \\ 0 & X(EX)^\dagger \tilde{A} B \end{pmatrix} \ge 0$$

because of Lemma 2(ii) and Remark 8(ii). Of course we have $Q_{N+1}^* = U_{N+1}^* X_{N+1}^{*-1}$, and this completes the proof for this implication.

Finally suppose that some Q^* satisfies the conditions in (iv). Let $(x, u) \in \mathscr{R}$ be admissible. Then we have, again by Remark 3(ii), that

$$R_k^*[Q^*]x_k^* = 0 \qquad \forall\, k \in J, \qquad \text{where } x^* := \begin{pmatrix} -x_0 \\ x \end{pmatrix},$$

and by applying Lemma 2(i) to the system (H*) we know (compare also the generalization of Picone's Identity, Proposition 4)

$$\begin{aligned} \mathscr{F}(x, u) = {} & \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T (S + Q_{N+1}^*) \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} - \begin{pmatrix} -x_0 \\ x_0 \end{pmatrix}^T Q_0^* \begin{pmatrix} -x_0 \\ x_0 \end{pmatrix} \\ & + \sum_{k=0}^{N} z_k^{*T} D_k^* z_k^* \\ = {} & c^T R(S + Q_{N+1}^*) R^T c + \sum_{k=0}^{N} z_k^{*T} D_k^* z_k^* \ge 0, \end{aligned}$$

where $\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} = R^T c$, and where $z^* = u^* - Q^* x^*$. If $\mathscr{F}(x, u) = 0$, then $D^* z^* = c = 0$ so that $x_0 = x_{N+1} = 0$ and hence $x_{N+1}^* = 0$. The last part of Lemma 2(i) shows as in (5) of Theorem 2 that $x = 0$, and the proof is complete. ∎

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